On the inverse Kostka matrix

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Abstract

In the ring of symmetric functions the inverse Kostka matrix appears as the transition matrix from the bases given by monomial symmetric functions to the Schur bases.

We present both a combinatorial characterization and a recurrent formula for the entries of the inverse Kostka matrix which are different from the results obtained by Egecioglu and Remmel [ER] in 1990. An application to the topology of the classifying space BU(n) is obtained.

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1 Introduction

By a *partition* we mean a sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of n non-negative integers in non-decreasing order $0 \le \lambda_1 \le \dots \le \lambda_n$. The following alternative notions for partitions will be useful in simplifying our presentation.

1) For a sequence $0 \le \lambda_1 \le \dots \le \lambda_k$ of k integers with $k \le n$, $\lambda = (\lambda_1, \dots, \lambda_k)$ stands for the partition that differs from λ only by $n-k$ of zero's at the beginning.

2) If $\lambda_1 \ge 1$ we may write $\lambda = \{\lambda_1, \dots, \lambda_k\}$ instead of $\lambda = (\lambda_1, \dots, \lambda_k)$.

3) Sometimes we use $\lambda = (r_1^{i_1}, \dots, r_k^{i_k})$, where

$1 \le r_1 < \dots < r_k$; $1 \le i_1, \dots, i_k$

and where $\Sigma i_s \le n$, to indicate that λ is the partition which begins with $n - \Sigma i_s$ copies of 0, followed by i_1 copies of r_1, then i_2 copies of r_2, …, etc.

Every partition can be uniquely written in the form of 2) (resp. of 3)). If this is the case, the number k (resp. Σi_s) will be called *the length of* λ and will be denoted by $l(\lambda)$.

We say λ is a partition of m if $m = \lambda_1 + \ldots + \lambda_n$. Denote by $P(m \mid n)$ for the set of all partitions of m. Assume throughout that $n \geq m$.

Let Λ_n be the ring of symmetric functions in the variables $x_1, \ldots, x_n$. It is graded by $\Lambda_n = \bigoplus_{m \geq 0} \Lambda_n^m$, where Λ_n^m is the $\mathbb{Z}$-module consisting of all homogenous symmetric polynomials of degree m, together with the zero polynomial. We recall two canonical $\mathbb{Z}$-bases of Λ_n^m, both are parameterized by elements in $P(m \mid n)$.

Let $\mathbb{N}^n$ be the set of n-tuples of non-negative integers. The set of partitions $P(m \mid n)$ will be considered as a subset of $\mathbb{N}^n$ in the obvious way. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ write x^α for the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

Let S_n be the permutation group on $\mathbb{N}^n$, and let $S_n(\alpha)$ be the stabilizer of S_n at $\alpha \in \mathbb{N}^n$. The set of left cosets of $S_n(\alpha)$ in S_n is denoted by S_n^α.

Definition. The *monomial symmetric functions* $m_\lambda(n)$ associated to a $\lambda \in P(m \mid n)$ is the element of Λ_n^m defined by $m_\lambda(n) = \sum_{w \in S_n^\lambda} x^{w(\lambda)}$.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ let $a_\alpha = \det(x_j^{\alpha_i})_{n \times n}$. The *Schur functions* $s_\lambda(n)$ associated to $\lambda = (\lambda_1, \ldots, \lambda_n) \in P(m \mid n)$ is the element of Λ_n^m defined by $s_\lambda(n) = a_{\lambda + \delta(n)} / a_{\delta(n)}$, where $\delta(n) = (0, 1, \ldots, n-1)$ and where the addition $\lambda + \delta(n)$ takes place in $\mathbb{N}^n$.$\square$

It is well known that both of the two sets of functions $\{m_\lambda(n) \mid \lambda \in P(m \mid n)\}$ and $\{s_\lambda(n) \mid \lambda \in P(m \mid n)\}$ constitute additive bases of Λ_n^m (cf. [Ma]). An immediate consequence is the existence of transition matrixes (with integer entries) from one to the other. Namely, we have

$$s_\lambda(n) = \sum_{\mu \in P(m|n)} K_{\lambda,\mu} m_\mu(n);$$

$$m_\lambda(n) = \sum_{\mu \in P(m|n)} K_{\lambda,\mu}^{-1} s_\mu(n) \tag{1.1}$$

for some integer matrices $K_m = (K_{\lambda,\mu})$, $K_m^{-1} = (K_{\lambda,\mu}^{-1})$ with $K_m K_m^{-1} = Id$.

The matrix K_m (resp. K_m^{-1}) is known as the *Kostka matrix* (resp. the *inverse Kostka matrix*). They were introduced by Kostka in 1882 [K1], who also computed K_m and K_m^{-1} up to $m = 11$ [K2]. Despite much combinatorial interest in these matrices (cf. [Ma, p.99-111]) we choose to emphasize a connection of a topological problem to the question of effective computations of certain $K_{\lambda,\mu}^{-1}$.

Let $BU(n)$ (resp. $BO(n)$) be the classifying space for the complex unitary group $U(n)$ (resp. the real orthogonal group $O(n)$) of order n, and let

$1+c_1+\ldots+c_n \in H^*(BU(n);\mathbb{Z})$ (resp. $1+w_1+\ldots+w_n \in H^*(BO(n);\mathbb{Z}_2)$) be the total Chern class of the universal complex n-bundle over $BU(n)$ (resp. the total Stiefel-Whitney class of the universal real n-bundle over $BO(n)$) [MS, §5]. Then

$$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1,\ldots,c_n] \quad (\text{resp. } H^*(BO(n);\mathbb{Z}_2) = \mathbb{Z}_2[w_1,\ldots,w_n]).$$

As is classically known, the correspondence $c_m \to m_{(1^m)}$ (resp. $w_m \to m_{(1^m)}$ mod 2) establishes a grading preserving isomorphism

$$H^*(BU(n);\mathbb{Z}) \cong \Lambda_n \text{ (resp. } H^*(BO(n);\mathbb{Z}_2) \cong \Lambda_n \otimes \mathbb{Z}_2) \text{ [MS, §7].}$$

In particular, the symmetric functions $s_\lambda(n)$ and $m_\lambda(n)$ (resp. module 2) can be interpreted as closed cocycles on $BU(n)$ (resp. on $BO(n)$) in which the $s_\lambda(n)$ act as the Kronecker dual of the classical Schubert varieties $\Omega(\lambda)$ in $BU(n)$ (resp. in $BO(n)$) [MS, §6].

Let $\mathcal{P}^k$ (resp. let Sq^k) be the Steenrod mod-p operations for a prime $p \neq 2$ on the mod-p (resp. the Steenrod mod-2 operations on the mod-2) cohomology of spaces [St]. From the Cartan formula [St] we get

$$\mathcal{P}^k(c_m) = m_{(1^{m-k},p^k)} \text{ mod } p \text{ in } H^*(BU(n);\mathbb{Z}_p)$$

$$(\text{resp. } Sq^k(w_m) = m_{(1^{m-k},2^k)} \text{ mod } 2 \text{ in } H^*(BO(n);\mathbb{Z}_2)),$$

where $k \leq m$. Combining this with (1.1) yields

$$(1.2) \quad \mathcal{P}^k(c_m) = \sum_{\mu\in P(m+(p-1)k|n)} K^{-1}_{(1^{m-k},p^k),\mu} s_\mu \text{ mod } p.$$

$$(\text{resp. } Sq^k(w_m) = \sum_{\mu\in P(m+k|n)} K^{-1}_{(1^{m-k},2^k),\mu} s_\mu \text{ mod } 2).$$

In view of the decomposition of $BU(n)$ given by Schubert cells, the geometric significance of (1.2) is clear: the number $K^{-1}_{(1^{m-k},p^k),\mu}$ gives information on how the cell $\Omega(\mu)$ is attached to $\Omega(1^m)$. We quote from Lenart [L]: very little is known about the attaching maps of Schubert cells.

In an attempt to find generalization of the famous Wu-formula [W]

$$(1.3) \qquad Sq^k(w_m) = \sum_{0\leq i\leq k} \binom{m-i-1}{k-i} w_i w_{m+k-i} \text{ (mod 2)},$$

many works were devoted to express $\mathcal{P}^k(c_m)$ as a polynomial in the Chern classes c_i (See Shay [S] or Lenart [L] for the history and relevant works on this). However, this task may alternatively be implemented by combining (1.2) with the Giambelli-formula [Ma] which expresses s_μ as a polynomial in the c_i.

Turning to the numerical aspects of the matrix K_m^{-1}, Egecioglu and Remmel [ER] obtained in 1990 a combinatorial interpretation and a recurrent formula for $K^{-1}_{\lambda,\mu}$. We briefly recall their result.

For a partition $\lambda = (r_1^{i_1}, \ldots, r_k^{i_k})$ and $1 \leq j \leq k$, let $\lambda[j]$ denote the partition which results from λ by removing a part r_j. That is $\lambda[j] = (r_1^{i_1}, \ldots, r_{j-1}^{i_{j-1}}, r_j^{i_j - 1}, r_{j+1}^{i_{j+1}} \ldots, r_k^{i_k})$.

If $\mu = \{\mu_1, \ldots, \mu_l\}$ is a partition and $1 \leq i \leq l$, we write $\omega \subset_i \mu$ for $\omega = (\mu_1 - 1, \ldots, \mu_{i-1} - 1, \mu_{i+1}, \ldots, \mu_l)$.

For two partitions $\lambda = \{\lambda_1, \ldots, \lambda_k\}, \mu = \{\mu_1, \ldots, \mu_l\} \in P(m \mid n)$ denote by $T(\lambda, \mu)$ the set of all chains of partitions

$T: \qquad \mu^0 = (0) \subset_{j_1} \mu^1 \subset_{j_2} \ldots \subset_{j_k} \mu^k = \mu$

so that if we have $\mu^i = \{\mu'_1, \ldots, \mu'_{l'}\}$ and put $a_i = \mu'_{j_i} + j_i - 1$ for each $1 \leq i \leq k$, then the sequence $(a_1, \ldots, a_k)$ is a permutation of $\lambda_1, \ldots, \lambda_k$.

For each $T \in T(\lambda, \mu)$ we set $sign(T) = (-1)^{j_1 + \ldots + j_k - k}$

Theorem 1 (Egecioglu-Remmel [ER]). *If* $\lambda = (r_1^{i_1}, \ldots, r_k^{i_k})$ *and* $\mu = \{\mu_1, \ldots, \mu_l\}$, *then*

$$(1.4) \qquad K^{-1}_{\lambda,\mu} = \sum_{\substack{1 \leq j \leq d \\ r_j = \mu_i + i - 1}} (-1)^{i-1} K^{-1}_{\lambda[j], (\mu_1 - 1, \ldots, \mu_{i-1} - 1, \mu_{i+1}, \ldots, \mu_l)}.$$

Consequently, $K^{-1}_{\lambda,\mu} = \sum_{T \in T(\lambda,\mu)} sign(T)$.

The merit of the recurrence (1.4) is apparent: without resorting to the symmetric functions (i.e. (1.1)), the computation of $K^{-1}_{\lambda,\mu}$ can be boiled down to the obvious relation $K^{-1}_{(0),(0)} = 1$.

It might be worthwhile to mention that Lenart announced in [L, Remark 5.4] also an algorithm to determine $K^{-1}_{\lambda,\mu}$.

In this paper we give another combinatorial description and recurrence for the numbers $K^{-1}_{\lambda,\mu}$. Our main results are stated in Section 2, followed by their proofs in Section 3. Our formula (2.1) is ready to apply to yield computational results. Examples are shown in Section 4.

2 Main results

For a $\lambda \in P(m \mid n)$ define a map $f_\lambda : S_n^\lambda \times S_n \to \mathbb{N}^n$ by letting

$$f_\lambda(w, \sigma) = w(\lambda) + \sigma(\delta(n)).$$

Let $\varepsilon : S_n \to \{\pm 1\}$ be the sign function on the permutation group S_n.

Theorem 2. *Assume that* $m_\lambda(n) = \sum_{\mu \in P(m|n)} K^{-1}_{\lambda,\mu} s_\mu(n)$. *Then*

$$K^{-1}_{\lambda,\mu} = \sum_{(w,\sigma) \in f_\lambda^{-1}(\mu + \delta(n))} \varepsilon(\sigma).$$

Remark 1. Theorem 2 offers another interpretation of the numbers $K^{-1}_{\lambda,\mu}$. Given two partitions $\lambda, \mu \in P(m \mid n)$, the set $f_\lambda^{-1}(\mu + \delta(n))$ is of interest for it consists of solutions $(w, \sigma) \in S_n^\lambda \times S_n$ to the vector equation

$$w(\lambda) + \sigma(\delta(n)) = \mu + \delta(n).$$

Let $I : S_n \to \mathbb{N}$ be the length function on the permutation group. From the subset $f_\lambda^{-1}(\mu + \delta(n)) \subset S_n^\lambda \times S_n$ one may derive a polynomial in t as

$$f_{\lambda,\mu}(t) = \sum_{(w,\sigma)\in f_\lambda^{-1}(\mu+\delta(n))} (-t)^{I(\sigma)}.$$

Clearly, one has $f_{\lambda,\mu}(-1) = Card\{f_\lambda^{-1}(\mu + \delta(n))\}$. On the other hand $f_{\lambda,\mu}(1) = K^{-1}_{\lambda,\mu}$ by Theorem 2.

For two partitions $\mu = (\mu_1, \ldots, \mu_n)$ and $\omega = (\omega_1, \ldots, \omega_n)$, write $\mu - \omega \in \{r\}$ to simplify the statement that each of the differences $\mu_i - \omega_i$ is either 0 or 1, and the cardinality of the set $\{i \mid \mu_i - \omega_i = 1\}$ is precisely r (In the standard terminology of partitions [Ma, p.4-5], $\mu - \omega \in \{r\}$ is equivalent to the statement that $\omega \subset \mu$ and that the skew diagram $\mu - \omega$ is a vertical r-strip). The notion $\omega <_r \mu$ is used to indicate that $\mu^{(n)} - \omega \in \{r\}$, where $\mu^{(n)} = (\mu_1, \ldots, \mu_{n-1})$.

For two partitions $\lambda = \{\lambda_1, \ldots, \lambda_k\}, \mu = (\mu_1, \ldots, \mu_n) \in P(m \mid n)$ denote by $S(\lambda, \mu)$ the set of all chains of partitions

$$S: \quad \mu^0 = (0) <_{j_1} \mu^1 <_{j_2} \cdots <_{j_k} \mu^k = \mu$$

so that if we assume $\mu^i = (\mu'_1, \ldots, \mu'_n)$ and put $b_i = \mu'_n + j_i$ for each $1 \le i \le k$, then the sequence $(b_1, \ldots, b_k)$ is a permutation of $\lambda_1, \ldots, \lambda_k$.

For a $S \in S(\lambda, \mu)$ we set $sign(S) = (-1)^{j_1+\cdots+j_k}$.

Theorem 3. *If* $\lambda = (r_1^{i_1}, \ldots, r_k^{i_k}), \mu = (\mu_1, \ldots, \mu_n) \in P(m \mid n)$, *then*

$$K^{-1}_{\lambda,\mu} = \sum_{r_j-\mu_n\ge 0} (-1)^{r_j-\mu_n} \sum_{\mu^{(n)}-\omega\in\{r_j-\mu_n\}} K^{-1}_{\lambda[j],\omega}. \tag{2.1}$$

Consequently, $K^{-1}_{\lambda,\mu} = \sum_{S\in S(\lambda,\mu)} sign(S)$.

Usually, a combinatorial identity is achieved whenever a given quantity is evaluated in two different ways. Combining (1.4) with (2.1) gives

Corollary 1(Identities in the inverse of Kostka matrix). *For any two partitions* $\lambda = (r_1^{i_1}, \ldots, r_k^{i_k})$ *and* $\mu = \{\mu_1, \ldots, \mu_l\}$ *we have*

$$\sum_{\substack{r_j-\mu_n\ge 0\\ \mu^{(n)}-\omega\in\{r_j-\mu_n\}}} (-1)^{r_j-\mu_n} K^{-1}_{\lambda[j],\omega} = \sum_{\substack{1\le j\le k\\ r_j=\mu_i+i-1}} (-1)^{i-1} K^{-1}_{\lambda[j],(\mu_1-1,\ldots,\mu_{i-1}-1,\mu_{i+1},\ldots,\mu_l)}.$$

The non-triviality of these identities can be easily seen from many of their specializations.

Example 1. Taking $\lambda = (2, 3)$ and $\mu = (1^3, 2)$ gives

$$K^{-1}_{(3),(1^3)} - K^{-1}_{(2),(1^2)} = -K^{-1}_{(3),(1,2)} + K^{-1}_{(2),(2)}.$$

If $\lambda = (1, 2^2)$ and $\mu = (1^3, 2)$ we have

$$K^{-1}_{(1,2),(1^3)} = K^{-1}_{(2^2),(1^2,2)} - K^{-1}_{(1,2),(1,2)}.$$

Remark 2. In the theory of symmetric functions the *Hall-Littlewood functions* (associated to a partition $\lambda \in P(m \mid n)$) is defined as

$$P_\lambda(x_1, \ldots, x_n; t) = \sum_{w \in S_n^\lambda} w(x^\lambda \prod_{\lambda_i < \lambda_j} \tfrac{x_j - t x_i}{x_j - x_i}) \text{ [Ma,p.208].}$$

Since P_λ is symmetric in $x_1, \ldots, x_n$ we have the expression

$$P_\lambda(x_1, \ldots, x_n; t) = \sum_{\mu} K^{-1}_{\lambda,\mu}(t) s_\mu$$

for some polynomial $K^{-1}_{\lambda,\mu}(t) \in \mathbb{Z}[t]$ (cf. [Ma, p.209]) which might be considered as one-parameter variation of $K^{-1}_{\lambda,\mu}$ in view of the fact $K^{-1}_{\lambda,\mu}(1) = K^{-1}_{\lambda,\mu}$, it would be of interest to see if Corollary 1 can be deduced from certain relations among the polynomials $K^{-1}_{\lambda,\mu}(t)$ (when evaluated at $t = 1$).

For $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n) \in P(m \mid n)$, write $\lambda > \mu$ to express the fact that the last non-zero difference $\lambda_i - \mu_i$ is positive.

Corollary 2 (Cancellation principles). *Given $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n) \in P(m \mid n)$, we have*

1) If $\lambda_j = \mu_j$ for $k \le j \le n$, then

$$K^{-1}_{(\lambda_1,\ldots,\lambda_n),(\mu_1,\ldots,\mu_n)} = K^{-1}_{(\lambda_1,\ldots,\lambda_k),(\mu_1,\ldots,\mu_k)}.$$

In particular, $K^{-1}_{\lambda,\lambda} = K^{-1}_{0,0} = 1$ for all $\lambda \in P(m \mid n)$.

2) If either $\lambda < \mu$ or $l(\lambda) > l(\mu)$, then $K^{-1}_{\lambda,\mu} = 0$.

Proof. If $\mu_n \ge \lambda_n$, we get immediately from (2.1) that

$$(2.2) \qquad K^{-1}_{\lambda,\mu} = \begin{cases} K^{-1}_{(\lambda_1,\ldots,\lambda_{n-1}),(\mu_1,\ldots,\mu_{n-1})}, & \text{if } \mu_n = \lambda_n; \\ 0, & \text{if } \mu_n > \lambda_n. \end{cases}$$

This verifies 1) and the first item of 2). The second assertion in 2) is also clear by Theorem 3, since the set $S(\lambda, \mu)$ must be empty if $l(\lambda) > l(\mu)$.□

3 Proofs of Theorem 2 and 3

Let $\mathbb{Z}[x_1, \ldots, x_n]$ be the ring of polynomials in $x_1, \ldots, x_n$ and, for an $m \ge 0$, let $\mathbb{Z}[x_1, \ldots, x_n]^m$ be the submodule spanned by the homogeneous polynomials of degree m. By considering Λ_n a subring of $\mathbb{Z}[x_1, \ldots, x_n]$ in the obvious way, Λ_n^m becomes a submodule of $\mathbb{Z}[x_1, \ldots, x_n]^m$.

For a sequence $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ consider the additive operator

$$D_\alpha : \mathbb{Z}[x_1, \ldots, x_n]^{|\alpha|} \to \mathbb{Z}, \mid \alpha \mid = \Sigma \alpha_i,$$

by $D_\alpha h =$the coefficient of the monomial x^α in h. Alternatively, it can be expressed in terms of partial derivatives as

$$D_\alpha h = \frac{1}{\alpha_1!\ldots\alpha_n!}\frac{\partial^{|\alpha|}h}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}.$$

In [D], [ZD] these operators were applied to express the integrations along certain flag manifolds and were useful in computing the degrees of some classical projective varieties.

Since $\{s_\lambda(n) \mid \lambda \in P(m \mid n)\}$ is an additive bases of Λ_n^m, every $h \in \Lambda_n^m$ can be uniquely written as an integral combination of the $s_\lambda(n)$. We give such an algorithm (cf. [D, Lemma 3.3]).

Lemma 1. If $h = \sum_{\lambda\in P(m|n)} c_\lambda s_\lambda(n)$, then $c_\lambda = D_{\lambda+\delta(n)}(ha_{\delta(n)})$.

Proof. For a $\mu = (r_1, \ldots, r_n) \in P(m \mid n)$ multiplying both sides of $h = \sum_{\lambda\in P(m|n)} c_\lambda s_\lambda$ by $a_{\delta(n)}$, and applying $D_{\mu+\delta(n)}$ yields

$$D_{\mu+\delta(n)}(ha_{\delta(n)}) = \sum_{\lambda\in P(m|n)} c_\lambda D_{\mu+\delta(n)}(a_{\lambda+\delta(n)}).$$

The proof is completed by observing that the monomial $x^{\mu+\delta(n)}$ is a term in $a_{\lambda+\delta(n)}$ if and only if $\lambda = \mu$ (note that the sequences $\lambda + \delta(n)$ and $\mu + \delta(n)$ are strict increasing), and its coefficient in $a_{\mu+\delta(n)}$ is precisely 1.□

Theorem 1 follows directly from Lemma 1.

Proof of Theorem 1. From Lemma 1 we have

$$K_{\lambda,\mu} = D_{\mu+\delta(n)}(m_\lambda(n)a_{\delta(n)})$$

$$= D_{\mu+\delta(n)}\Big(\sum_{w\in S_n^\lambda} x^{w(\lambda)} \sum_{\sigma\in S_n} \varepsilon(\sigma)x^{\sigma(\delta(n))}\Big) = \sum_{\substack{(w,\sigma)\in S_n^\lambda\times S_n \\ w(\lambda)+\sigma(\delta(n))=\mu+\delta(n)}} \varepsilon(\sigma).\square$$

The proof of Theorem 2 requires a little more preparation. The first of these is the following "elimination law"(cf. [D, Section 3]).

Lemma 2. *If* $h(x_1, \ldots, x_n) = \Sigma h_i x_n^i$, *with* $h_i \in Z[x_1, \ldots, x_{n-1}]$, *then*

$$D_{(r_1,\ldots,r_n)}h(x_1,\ldots,x_n) = D_{(r_1,\ldots,r_{n-1})}h_{r_n}(x_1,\ldots,x_{n-1}).$$

Let $e_r(n) \in \Lambda_n^r$ be the r^{th} elementary symmetric function, $0 \leq r \leq n$. Then $e_r(n) = m_{(1^r)}(n)$. The next result (cf. [Ma, p.73]) is known as the *Pieri-formula*.

Lemma 3. *For any* $\lambda \in P(m \mid n)$, $s_\lambda(n)e_r(n) = \sum_{\mu-\lambda\in\{r\}} s_\mu(n)$.

The correspondence $P(m \mid n-1) \to P(m \mid n)$ by $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \to \lambda' = (0, \lambda_1, \ldots, \lambda_{n-1})$ is a bijection whenever $m \leq n-1$ [Ma, p.18]. The stability of the numbers $K_{\lambda,\mu}^{-1}$ can now be stated as

Lemma 4. *If* $\lambda, \mu \in P(m \mid n-1)$ *and if* $m \leq n-1$, *then* $K_{\lambda,\mu}^{-1} = K_{\lambda',\mu'}^{-1}$.

In view of this we do not differentiate between λ and λ'.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in P(m \mid n)$ and for $1 \le i \le n$, write $\lambda^{(i)}$ for the partition $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}+1, \ldots, \lambda_n+1) \in P(m' \mid n-1)$, where $m' = m - \lambda_i + (n-i)$.

Proof of Theorem 2. Suppose that for all $m \le n$ and $\lambda, \mu \in P(m \mid n)$ the numbers $K^{-1}_{\lambda,\mu}(n) \in \mathbb{Z}$ are defined by

$$m_\lambda(n) = \sum_{\mu \in P(m|n)} K^{-1}_{\lambda,\mu}(n) s_\mu(n) \tag{3.1}$$

With $\lambda = (r_1^{i_1}, \ldots, r_k^{i_k})$, the formula expressing $m_\lambda(n)$ as a polynomial in x_n with coefficients from $\mathbb{Z}[x_1, \ldots, x_{n-1}]$ is

$$m_\lambda(n) = m_\lambda(n-1) + \sum_{1 \le j \le k} m_{\lambda[j]}(n-1) x_n^{r_j}.$$

Expanding the determinant $a_{\delta(n)}$ with respect to the last column yields

$$a_{\delta(n)} = \sum_{1 \le i \le n} (-1)^{i-1} a_{\delta(n)^{(n-i+1)}} x_n^{n-i}.$$

It follows that

$$m_\lambda(n) a_{\delta(n)} = m_\lambda(n-1) a_{\delta(n)} + \sum_{\substack{1 \le i \le n \\ 1 \le j \le k}} (-1)^{i-1} m_{\lambda[j]}(n-1) a_{\delta(n)^{(n-i+1)}} x_n^{r_j+n-i}.$$

¿From Lemma 1 we have

$$K^{-1}_{\lambda,\mu}(n) = D_{\mu+\delta(n)}(m_\lambda(n) a_{\delta(n)}). \tag{3.2}$$

Since we can assume $\mu_n \ge 1$, the coefficient of $x_n^{\mu_n+n-1}$ in $m_\lambda(n-1)a_{\delta(n)}$ is clearly zero. Applying the elimination law (Lemma 2) to the right hand side of (3.2) gives

$$\begin{aligned} K^{-1}_{\lambda,\mu}(n) &= D_{\mu^{(n)}+\delta(n-1)}\Big(\sum_{\substack{\mu_n+n-1=r_j+n-i \\ 1 \le j \le k}} (-1)^{i-1} m_{\lambda[j]}(n-1) a_{\delta(n)^{(n-i+1)}} \Big) \\ &= D_{\mu^{(n)}+\delta(n-1)}\{ [\sum_{\substack{i=r_j+1-\mu_n \\ 1 \le j \le k}} (-1)^{i-1} m_{\lambda[j]}(n-1) e_{i-1}(n-1)] a_{\delta(n-1)} \}, \end{aligned} \tag{3.3}$$

where $\mu^{(n)} = (\mu_1, \ldots, \mu_{n-1}) (\in P(m-\mu_n \mid n-1))$, and where the second equality follows from $a_{\delta(n)^{(n-i+1)}} = e_{i-1}(n-1) a_{\delta(n-1)}$. Since

$$m_{\lambda[j]}(n-1) = \sum_{\omega \in P(m-r_j|n-1)} K^{-1}_{\lambda[j],\omega}(n-1) s_\omega(n-1)$$

by our assumption (3.1) and since

$$s_\omega(n-1) e_{i-1}(n-1) = \sum_{\gamma - \omega \in \{i-1\}} s_\gamma(n-1)$$

by Lemma 3, we get from (3.3) that

$$K^{-1}_{\lambda,\mu}(n) = \sum_{r_j - \mu_n \ge 0} (-1)^{r_j-\mu_n} \sum_{\mu^{(n)} - \omega \in \{r_j - \mu_n\}} K^{-1}_{\lambda[j],\omega}(n-1)$$

(again by Lemma 1). Finally, $\lambda[j], \omega \in P(m - r_j \mid n-1)$ implies that

$$K^{-1}_{\lambda,\mu}(n) = \sum_{r_j - \mu_n \ge 0} (-1)^{r_j-\mu_n} \sum_{\mu^{(n)} - \omega \in \{r_j - \mu_n\}} K^{-1}_{\lambda[j],\omega}(n)$$

by Lemma 4. This completes the proof.□

4 Applications

Theorem 3 (i.e. (2.1)) enables one to deduce explicit formulas for $K^{-1}_{\lambda,\mu}$ for special cases of λ and μ. This section is devoted to such examples that have interesting numerical features.

Consider firstly $\lambda = (r_1^{i_1}, \ldots, r_k^{i_k})$, $\mu = (1^m) \in P(m \mid n)$. We have

$$K^{-1}_{\lambda,(1^m)} = \sum_{1\le j\le k} (-1)^{r_j-1} K^{-1}_{\lambda[j],(1^{m-r_j})}. \tag{4.1}$$

by (2.1). An induction on $l(\lambda) = \Sigma i_j$ verifies (cf. [ER, Corollary 1]):

Lemma 5. $K^{-1}_{\lambda,(1^m)} = \frac{(-1)^{l(\mu)-l(\lambda)} l(\lambda)!}{i_1! i_2! \ldots i_k!}$.

Proof. $K^{-1}_{(m),(1^m)} = (-1)^{m-1}$ by Theorem 3. From the inductive hypothesis one has

$$K^{-1}_{\lambda[j],(1^{m-r_j})} = \frac{(-1)^{m-r_j-l(\lambda)-1}(l(\lambda)-1)!}{i_1! \ldots i_{j-1}!(i_j-1)! i_{j+1}! \ldots i_k!},\ 1 \le j \le k.$$

Substituting these in (4.1) yields

$$K^{-1}_{\lambda,(1^m)} = \frac{(-1)^{m-l(\lambda)}(l(\lambda)-1)!}{i_1! i_2! \ldots i_k!} \Sigma i_j = \frac{(-1)^{l(\mu)-l(\lambda)} l(\lambda)!}{i_1! i_2! \ldots i_k!}. \Box$$

For a $\lambda \in P(m \mid n)$ we set $\lambda'_i = Card\{j \mid \lambda_j \ge i\}$. Continuing from Lemma 5 we recover the following result originally due to Kostka [K2]

Lemma 6. $K^{-1}_{\lambda,(1^m a)} = \frac{(-1)^{l(\mu)-l(\lambda)}(l(\lambda)-1)!}{i_1! i_2! \ldots i_k!} \lambda'_a$.

Proof. $K^{-1}_{\lambda,(1^m a)} = \sum_{r_j \ge a} (-1)^{r_j-a} K^{-1}_{\lambda[j],(1^{m-r_j+a})}$ (by (2.1))

$$= \sum_{r_j \ge a} (-1)^{r_j-a} \frac{(-1)^{m-r_j+a-l(\lambda)+1}(l(\lambda)-1)!}{i_1! \ldots i_{j-1}!(i_j-1)! i_{j+1}! \ldots i_k!} \text{ (by Lemma 5)}$$

$$= \frac{(-1)^{l(\mu)-l(\lambda)}(l(\lambda)-1)!}{i_1! i_2! \ldots i_k!} \lambda'_a. \Box$$

An extension of Lemma 6 is the case $\mu = (1^m, a, b)$, $1 < a \le b$. We may assume that $\lambda = (r_1^{i_1}, \ldots, r_k^{i_k}) \ge \mu$ by 2) of Corollary 2. Then there exists a unique $d \le k$ such that $r_d \ge b$ but $r_i < b$ for all $1 \le i < d$. Consider (for instance) the case $r_d > b$. We have

$$K^{-1}_{\lambda,(1^m,a,b)} = \sum_{d\le j\le k} (-1)^{r_j-b} [K^{-1}_{\lambda[j],(1^{m-r_j+b},a)} + K^{-1}_{\lambda[j],(1^{m-r_j+b+1},a-1)}] \text{ (by (2.1))}$$

$$= \sum_{d\le j\le k} (-1)^{r_j-b} [\frac{(-1)^{m-r_j+b-l(\lambda)+1}(l(\lambda)-2)!}{i_1! \ldots i_{j-1}!(i_j-1)! i_{j+1}! \ldots i_k!} (\lambda[j]'_a - \lambda[j]'_{a-1})] \text{(by Lemma 6)}$$

$$= \frac{(-1)^{l(\mu)-l(\lambda)}(l(\lambda)-2)!}{i_1! i_2! \ldots i_k!} \sum_{d\le j\le k} i_j (\lambda[j]'_{a-1} - \lambda[j]'_a).$$

Thus, for a $\omega = (\omega_1, \ldots, \omega_k)$, if we let $D_c(\omega) = Card\{i \mid \omega_i = c\}$, we have

Corollary 3. *If $r_d > b$, then*

$$K^{-1}_{\lambda,(1^m,a,b)} = \frac{(-1)^{l(\mu)-l(\lambda)}(l(\lambda)-2)!}{i_1! i_2! \ldots i_k!} \sum_{d\le j\le k} i_j D_{a-1}(\lambda[j]);$$

If $r_d = b$, *then*

$$K^{-1}_{\lambda,(1^m,a,b)} = \frac{(-1)^{l(\mu)-l(\lambda)}(l(\lambda)-2)!}{i_1!i_2!\dots i_k!}[i_d\lambda[d]'_a + \sum_{d+1\le j\le k} i_j D_{a-1}(\lambda[j])].$$

Let us compute $K^{-1}_{(1^k,2^l),\mu}$. Firstly $K^{-1}_{(1^k,2^l),\mu} = 0$ if $\mu \neq (1^{k+2t}, 2^{l-t})$ by 2) of Corollary 2. Moreover

$$K^{-1}_{(1^k,2^l),(1^{k+2t},2^{l-t})} = K^{-1}_{(1^k,2^t),(1^{k+2t})} \text{ (by 1) of Corollary 2)}$$
$$= \frac{(-1)^t(k+t)!}{k!t!} \text{ (By Lemma 5).}$$

Summarizing we get (cf. also [ER, Corollary 3])

Corollary 4. $m_{(1^k,2^l)}(n) = \sum_{0\le t\le l} \frac{(-1)^t(k+t)!}{k!t!} s_{(1^{k+2t},2^{l-t})}(n).$

Remark 3. Combining Corollary 4 with the Giambelli-formula

$$s_{(1^{m-k},2^k)}(n) = e_k(n)e_m(n) - e_{k-1}(n)e_{m+1}(n)$$

yields an integral lift of the Wu-formula (1.3).□

Let us compute $K^{-1}_{(1^k,3^l),\mu}$. Again, By 2) of Corollary 2 we have $K^{-1}_{(1^k,3^l),\mu} = 0$ unless $\mu = (1^a, 2^b, 3^c)$ with $c \le l$. Furthermore, 1) of Corollary 2 implies that $K^{-1}_{(1^k,3^l),(1^a,2^b,3^c)} = K^{-1}_{(1^k,3^{l-c}),(1^a,2^b)}$. So it remains to find $K^{-1}_{(1^k,3^l),(1^a,2^b)}$, where $k + 3l = a + 2b$.

For fixed k and l consider the polynomial in t

$$g_{k,l}(t) = \sum_{0\le b, k+3l=a+2b} K^{-1}_{(1^k,3^l),(1^a,2^b)} t^b. \tag{4.2}$$

Since

$$K^{-1}_{(1^k,3^l),(1^a,2^b)} = \begin{cases} \frac{(k+l)!}{k!l!} \text{ if } b = 0,\ a \ge 1 \text{ (Lemma 5);} \\ -K^{-1}_{(1^k,3^{l-1}),(1^{a-1})} \text{ if } b = 1, a \ge 1 \text{ (by (2.1));} \\ -K^{-1}_{(1^k,3^{l-1}),(1,2^{b-2})} \text{ if } a = 0 \text{ (by (2.1)),} \end{cases}$$

and since

$$K^{-1}_{(1^k,3^l),(1^a,2^b)} = -K^{-1}_{(1^k,3^{l-1}),(1^{a-1},2^{b-1})} - K^{-1}_{(1^k,3^{l-1}),(1^{a+1},2^{b-2})}$$

if $b \ge 2$, $a \ge 1$ (by (2.1)), we obtain

$$g_{k,l}(t) = \frac{(k+l)!}{k!l!} - (t+t^2)g_{k,l-1}(t) + \varepsilon(t), \tag{4.3}$$

where

$$\varepsilon(t) = \begin{cases} 0 & \text{if } k+3l \text{ is even;} \\ K^{-1}_{(1^k,3^l),(2^{\frac{k+3(l-1)}{2}})} t^{\frac{k+3(l-1)}{2}} & \text{if } k+3l \text{ is odd.} \end{cases}$$

Assume now that $k > l-1$. Then $\varepsilon(t) = 0$ by 2) of Corollary 2. We infer from (4.3) and the obvious relation $g_{k,0}(t) = (-1)^{k-1}$ (by Lemma 5) that

Corollary 5. *If* $k > l-1$, *then* $g_{k,l}(t) = \sum_{0\le i\le l} (-1)^i \frac{(k+l-i)!}{k!(l-i)!}(t+t^2)^i.$

In order to apply (4.3) to find an expression of $g_{k,l}(t)$ for the case $k \leq l-1$, we need to compute $K^{-1}_{(1^k,3^l),(2^b)}$ (for $k+3l$ even). This will be done by combining (2.1) with the Egecioglu-Remmel formula (1.4). Consider, for a fixed b, the polynomial in t

$$h_b(t) = \sum_{k+3l=2b} K^{-1}_{(1^k,3^l),(2^b)} t^k.$$

From

$$K^{-1}_{(1^k,3^l),(2^b)} = -K^{-1}_{(1^k,3^{l-1}),(1,2^{b-2})} \text{ (by (2.1))}$$
$$= \{ \begin{array}{l} -K^{-1}_{(1^{k-1},3^{l-1}),(2^{b-2})} + K^{-1}_{(1^k,3^{l-2}),(2^{b-3})} \text{ if } k \geq 1; \\ K^{-1}_{(3^{l-2}),(2^{b-3})} \text{ if } k=0 \end{array}$$

(by (1.4)) we get

$$h_b(t) = -th_{b-2}(t) + h_{b-3}(t).$$

It follows that

$$h_b(t) = (t^2, -2t, 1) \begin{pmatrix} h_{b-4} \\ h_{b-5} \\ h_{b-6} \end{pmatrix}$$

and that

$$\begin{pmatrix} h_{b-4} \\ h_{b-5} \\ h_{b-6} \end{pmatrix} = A(t) \begin{pmatrix} h_{b-6} \\ h_{b-7} \\ h_{b-8} \end{pmatrix}, \text{ where } A(t) = \begin{pmatrix} -t & 1 & 0 \\ 0 & -t & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Corollary 6. *Assume that $b = 2k + r$ with $0 \leq r \leq 1$. Then*

$$h_b(t) = (t^2, -2t, 1) A(t)^{k-3} \begin{pmatrix} h_{2+r}(t) \\ h_{1+r}(t) \\ h_r(t) \end{pmatrix},$$

where $(h_0(t), h_1(t), h_2(t), h_3(t)) = (1, 0, -t, 1)$.

The last equation can be easily obtained by using either (1.4) or (2.1).

Example 2. Based on Corollary 6 a program to expand $h_b(t)$ has been compiled. We list below the results for $25 \leq b \leq 30$ produced by the program.

$$h_{25}(t) = 36t^2 - 252t^5 + 165t^8 - 12t^{11};$$
$$h_{26}(t) = -9t + 210t^4 - 330t^7 + 66t^{10} - t^{13};$$
$$h_{27}(t) = 1 - 120t^3 + 462t^6 - 220t^9 + 13t^{12};$$
$$h_{28}(t) = 45t^2 - 462t^5 + 495t^8 - 78t^{11} + t^{14};$$
$$h_{29}(t) = -10t + 330t^4 - 792t^7 + 286t^{10} - 14t^{13};$$
$$h_{30}(t) = 1 - 165t^3 + 924t^6 - 715t^9 + 91t^{12} - t^{15}.$$

By the discussion in Section 1 on the Steenrod cohomology operations, the above polynomials can be used to reveal deep information on the topology of the classifying space $BU(n)$ (as well as the complex Grassmannians). For instance, we see from

$$h_{30}(t) \equiv 1 + 2t^9 + t^{12} + 2t^{15} \pmod 3$$

that the attaching map of the Schubert cell $\Omega(2^{30})$, which has the real dimension 120, cannot avoid any of the Schubert cells $\Omega(1^{10})$, $\Omega(1^{16})$, $\Omega(1^{18})$ and $\Omega(1^{20})$ via homotopies.

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